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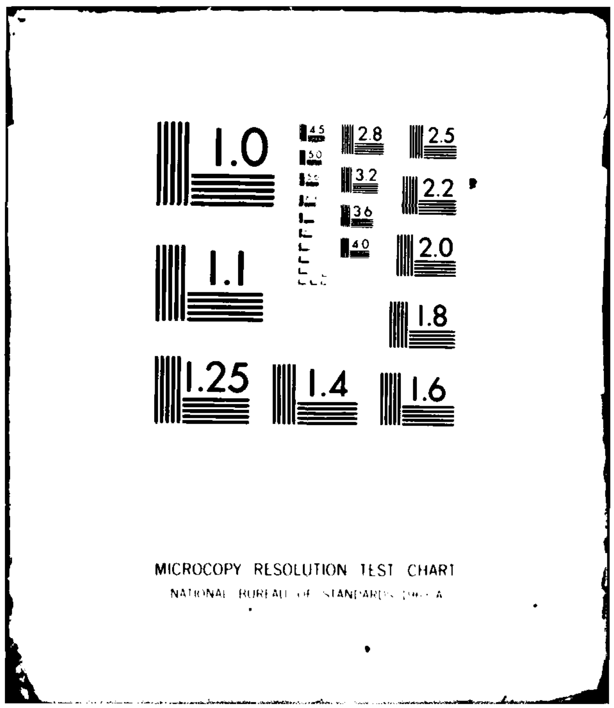
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ON CHARACTERIZING THE MARKOV POLYA DISTRIBUTION

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University of Pittsburgh

February 1981

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*On sabbatical leave from Sangamon State University, Springfield, IL 62708

ON CHARACTERIZING THE MARKOV POLYA DISTRIBUTION

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ABSTRACT

A discrete survival model is considered where an unobservable random variable is subjected to destruction so that what is observed and recorded is only the undamaged part X of N. Assuming the destruction process is represented by the Markov-Polya distribution, a characterization of the negative binomial distribution is obtained. Utilizing the completeness property of the negative binomial distribution, a characterization of the Markov-Polya distribution is derived. Several other characterization theorems are also proved concerning these probability distributions.

Key Words: characterization; Markov-Polya distribution; negative binomial; damage model; survival distribution; birth process; completeness of families.

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1. INTRODUCTION

Contagious distributions are characteristic of many biological and ecological systems. Numerous distributions are developed in statistical and biological literature for dealing with certain contagious events (see for example, Johnson and Kotz, 1969 chp. 9 and the references contained there in). The Markov-Polya distribution, which was derived first by Markov (1917) and later by Polya in Eggenberger and Polya (1923), is one such distribution. The genesis of this distribution is generally presented in terms of random drawings of balls from an urn. Initially, it is assumed that there are $\underline{a}$ white balls and, $\underline{b}$ black balls in the urn. One ball is drawn at random and then replaced with $\underline{c}$ additional balls of the same color. This procedure is repeated N times. Then the total number, X, of the white balls in the sample will have the Markov-Polya distribution (MPD). A vast number of interesting properties of this distribution have been discussed by Bosch (1963) and Dyczka (1972) among others. However, characterizations of this distribution, to the best of this author's knowledge, are not discussed in the literature. Thus, the purpose of this paper is to present certain important and interesting characterization theorems concerning the MPD.

First of all, the MPD is generated, in section 2, from a pure birth process. In section 3, a characterization of the MPD is provided utilizing the reproducible property of the NBD. Two characterization theorems, analogous to the theorems of Rao-Rubin (1964), based on the concept of damage models are given in section 4, and section 5 provides four other theorems involving the MPD and the NBD based on conditional distribution concept.

2. PRELIMINARIES

Let the integer valued random variable, N, denote the size of a biological community which produces two types of children, say boys and girls for the sake of simplicity. Let X and Y denote the number of boys and girls respectively, where $N = X+Y$. If the accent is on X, we say that N is reduced to X by means of the Markov-Polya survival model:

$$S(k/n) = P(X=k|N=n) = \binom{n}{k} a^{(k,c)} b^{(n-k,c)}/(a+b)^{(n,c)} \qquad (2.1)$$

where $a>o$, $b>o$, $c\neq o$ and $k=o,1,2,\ldots,n$. If c is negative, then $(-c)(n-1) < \min(a,b)$. The expression $y^{(m,c)}$ is a factorial polynomial of the m-th degree with respect to y, which is given by

$$\begin{aligned} y^{(m,c)} &= y(y+c)(y+2c)\ldots(y+(m-1)c) \\ &= c^m\,\Gamma(m+y/c)/\Gamma(y/c) \end{aligned} \qquad (2.2)$$

The probability distribution (p.d.) can be generated by a discrete-time, discrete-state pure birth process as follows.

Let the initial probability of producing a boy be $a/(a+b)$. Let the probability of producing a boy change during the growth of the community size so that after having produced t children, of which k are boys, the probability that the next offspring is a boy is $(a+kc)/(a+b+tc)$. The constant $\underline{c}$ is interpreted as a parameter of contagion. Let $X(t)=k$ represent the number of boys out of t children. Then the transition probabilities are:

$$P[X(t+1)=k+1|X(t)=k] = (a+kc)/(a+b+tc), \qquad (2.3)$$

where $a>o$, $b>o$, $c\neq o$, and t is a positive integer. From the law of total probability, we can write the unconditional probability $P[X(t+1)=k+1]$ as

$$P[X(t+1) = k+1] =$$

$$P[X(t+1)=k+1 \mid X(t)=k]\, P[X(t)=k] + P[X(t+1)=k+1 \mid X(t)=k+1]\, P[X(t)=k+1] , \qquad (2.4)$$

for $k=0,1,2,\ldots,t$. If we denote $p_k(t) = P[X(t)=k]$, using (2.3) we can write (2.4) as

$$p_{k+1}(t+1) = \frac{a+kc}{a+b+tc}\, p_k(t) + \left[1 - \frac{a+(k+1)c}{a+b+tc}\right] p_{k+1}(t) , \qquad (2.5)$$

and

$$p_o(t+1) = \frac{a}{a+b+tc}\, p_o(t) \text{ with } p_o(o) = 1 . \qquad (2.6)$$

By solving the recurrence relations (2.5) and (2.6), we can easily see that

$$p_k(t) = \binom{t}{k} a^{(k,c)}\, b^{(t-k,c)} / (a+b)^{(t,c)}. \qquad (2.7)$$

A convenient and simple form of equation (2.1) is obtained by letting $p=a/(a+b)$, $q=b/(a+b)$ and $r=c/(a+b)$, which gives

$$P(k|n) = \binom{n}{k} p^{(k,r)}\, q^{(n-k,r)} / 1^{(n,k)} . \qquad (2.8)$$

where $o < p$, $q < 1$, $p+q=1$ and $r \neq o$.

Many standard discrete distributions can be obtained as special cases or as limiting cases of the MPD. For instance

i) $c = o$ gives the binomial distribution

ii) $c = -1$ gives the hypergeometric distribution

iii) $c = +1$ (with a,b positive integers) gives the negative hypergeometric distribution.

iv) The MPD (2.8) tends to the negative binomial distribution (NBD) with parameters h/β and $(1+\beta)^{-1}$ as $n\to\infty$, $p\to o$ and $r\to o$ such that $np\to h$ and $nr\to\beta$. This limiting distribution is sometimes called

the Polya-Eggenberger distribution. Because of its importance in the sequel, we define the NBD as follows:

Definition: A discrete r.v. N is said to have a negative binomial distribution with parameters k and p if its probability function (p.f.) is given by

$$f(n) = P(N=n) = \frac{\Gamma(n+k)}{n!\,\Gamma(k)}\, p^k q^n, \quad n=0,1,2,\ldots \tag{2.9}$$

On occasion, we denote this p.f. by $nb(n;k,p)$.

3. CHARACTERIZATION OF MPD IN TERMS OF REPRODUCIBILITY OF NBD

The following theorem gives a characterization of the MPD in terms of the reproducibility of the NBD.

Theorem 1: Consider the family of distributions $\phi(k/n)$ indexed by the parameter $n=0,1,2,\ldots$, and each supported on a subset of $\{k: k=0,1,\ldots,n\}$ and independent of θ. Let n follow an NBD, $nb(n;(a+b)/c,\theta)$. Then the resultant mixture distribution is an NBD, $nb(k;a/c,\theta)$ if, and only if, $\phi(k|n) \equiv s(k|n)$. That is, $s(k|n)$ as given in (2.1) is the unique solution of

$$\sum_{n=0}^{\infty} s(k|n)\, nb(n;(a+b)/c,\theta) = nb(k;a/c,\theta) \tag{3.1}$$

Proof: Sufficiency.

In (3.1), let $s(k|n)$ be given by the MPD, then the left side of (3.1) is

$$\sum_{n=0}^{\infty} \binom{n}{k} \frac{a^{(k,c)} b^{(n-k,c)}}{(a+b)^{(n,c)}} \; \frac{(a+b)^{(n,c)}}{n!\, c^n} \; \theta^{(a+b)/c} (1-\theta)^n$$

$$= \frac{a^{(k,c)} \theta^{a/c} (1-\theta)^k}{k!\, c^k} \left[\sum_{n=k}^{\infty} \frac{b^{(n-k,c)} \theta^{b/c} (1-\theta)^{n-k}}{(n-k)!\, c^{n-k}} \right]$$

The sum of the terms in brackets is one. Thus, the distribution of the mixture is nb(k,a/c,θ).

Necessity.

That s(k/n) is the unique solution of (3.1) may be proved in various ways. A convenient one makes use of the concept of completeness of a family of distributions. Let us recall the concept of completeness of a family of distributions: $P = \{P_\theta : \theta \in \Omega\}$ of a r.v. X, indexed by the parameter set Ω, is complete if, for any function U(X) independent of θ, E[U(X)] = 0 for every θ ε Ω implies U(X) = 0 for all x (except possibly for a set of x with probability measure zero for all θ ε Ω).

We know that s(k/n) satisfies (3.1). Suppose that some other distribution g(k/n) also satisfy (3.1). We thus have

$$\sum_{n=0}^{\infty} s(k/n) \frac{\Gamma(n+(a+b)/c)}{n!\ \Gamma((a+b)/c)}\ \theta^{(a+b)/c}(1-\theta)^n =$$

$$\frac{\Gamma(k+a/c)}{k!\Gamma(a/c)}\ \theta^{a/c}(1-\theta)^k \qquad (3.2)$$

$$\sum_{n=0}^{\infty} g(k/n) \frac{\Gamma(n+(a+b)/c)}{n!\Gamma((a+b)/c)}\ \theta^{(a+b)/c}(1-\theta)^n =$$

$$\frac{\Gamma(k+a/c)}{k!\Gamma(a/c)}\ \theta^{a/c}(1-\theta)^k\ , \qquad (3.3)$$

for any fixed k.

Subtracting (3.3) from (3.2) we get

$$\sum_{n=0}^{\infty} U(n) \frac{\Gamma(n+(a+b)/c)}{n!\ \Gamma((a+b)/c)}\ \theta^{(a+b)/c}(1-\theta)^n = 0\ ,$$

where U(n) = s(k/n) - g(k/n). But it is well known that the negative binomial is complete. Hence, E[U(n)] = 0 implies U(n) = 0 and hence

$g(k/n) = s(k/n)$.

Remark: A multivariate extension of this theorem can easily be stated and proved.

4. CHARACTERIZATIONS BASED ON DAMAGE MODEL

Let (N,X) be a random vector of non-negative integer-valued components such that

$$P(N=n, X=k) = f(n)s(k/n), \tag{4.1}$$

where $\{f(n): n=0,1,2,\ldots\}$ and $\{s(k/n): k=0,1,2,\ldots,n\}$ for each $n \geq 0$ are discrete probability distributions. That is, the marginal distribution of N is $\{f(n)\}$ and for each $n \geq 0$ with $f(n) > 0$, the conditional distribution of X given $N=n$ is $\{s(k/n): k=0,1,2,\ldots,n\}$. Further,

$$P(X=k) = \sum_{n=k}^{\infty} f(n)s(k/n), \tag{4.2}$$

$$P\{X=k \mid \text{no damage}\} = \frac{f(n)s(k/k)}{\sum_{j=0}^{\infty} f(n)s(j/j)} \tag{4.3}$$

$$P\{X=k \mid \text{damaged}\} = \frac{\sum_{n=k+1}^{\infty} f(n)s(k/n)}{\sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty} f(n)s(k/n)} \tag{4.4}$$

Theorem 2: If a r.v. N defined on non-negative integers is distributed in nature as an NBD (2.2), with parameters $(k=(a+b)/c,\ p)$ and if it is damaged and reduced to X by the Markov-Polya survival model (2.1) and further, if Y is the resultant r.v., then

i) $P(X=k) = P(X=k \mid \text{damaged}) = P(X=k \mid \text{no damage})$, and

ii) Y has an NBD with parameters $(a/c, p)$.

Proof: Applying the equations (4.1), (4.2) and (4.3) the results (i) and (ii) follow.

<u>Theorem 3</u>: Let N be a non-negative integer valued r.v. and let the probability that an observation n of N is reduced to k during a destructive process be given by the Markov-Polya process (2.8). Let X be the resultant r.v. Then the necessary and sufficient condition that N has an NBD is that

$$P(X=k) = P(X=k \mid \text{damaged}) = P(X=k \mid \text{undamaged}). \qquad (4.5)$$

<u>Proof</u>: Necessary part follows from theorem 2. To prove sufficiency we observe that the condition (4.5) yields

$$\sum_{n=k}^{\infty} f(n)\binom{n}{k} p^{(k,r)} q^{(n-k,r)} / 1^{(n,r)} =$$

$$\frac{f(k)\, p^{(k,r)} / 1^{(k,r)}}{\sum_{j=0}^{\infty} f(j)\, p^{(j,r)} / 1^{(j,r)}} \qquad (4.6)$$

Define

$$f(y) = F(y)\, 1^{(y,r)}\, W^y/y! \quad , \qquad (4.7)$$

where W is some arbitrary quantity which will be determined later. Substituting (4.7) into (4.6) with $n = k+s$ and cancelling $p^{(k,r)}/k!$ on both sides of (4.6) we get

$$\sum_{s=0}^{\infty} F(k+s)\, q^{(s,r)}\, W^{k+s}/s! = \frac{W^k F(k)}{\sum_{j=0}^{\infty} F(j)\, p^{(j,r)}\, W^j/j!}$$

Let $G(pt) = \sum_{j=0}^{\infty} F(j)\, (pt)^{(j,r)}\, W^j/j!$,

where $G(0) = F(0)$. Multiplying the equation (4.8) by $(pt)^{(k,r)}/k!$ and summing over k from 0 to ∞, it becomes

$$\sum_{n=0}^{\infty} \frac{F(n)}{n!} \left[\sum_{j=0}^{n} \binom{n}{k} (pt)^{(k,r)} q^{(n-k,r)} \right] V^n = \frac{G(pt)}{G(p)} \tag{4.9}$$

Since the quantity in the square brackets is equal to $(pt+q)^{(n,r)}$, (4.9) becomes

$$G(pt+q)\, G(p) = G(pt). \tag{4.10}$$

Putting $u = p(t-1)$, (4.10) reduces to

$$G(u+1)\, G(u) = G(u+p).$$

Let $\phi(u) = G(u+1)$ and $v = p-1$, this reduces to Cauchy functional equation

$$\phi(u)\, \phi(v) = \phi(u+v)$$

the solution of which is given by $\phi(x) = e^{\lambda x}$. Hence $G(x) = e^{\lambda(x-1)}$.

Thus,

$$e^{\lambda p} = \sum_{j=0}^{\infty} F(j)\, e^{\lambda}\, p^{(j,r)}\, V^j/j! \tag{4.11}$$

Setting $e^{\lambda} = (1-\theta)^{-1/r}$ and $V = \theta/r$ in (4.11) we get

$$(1-\theta)^{-p/r} = \sum F(j)(1-\theta)^{-1/r}\, p^{(j,r)} (\theta/r)^j/j! \tag{4.12}$$

In order to determine $F(j)$, we consider the identity

$$(1-\theta)^{-p/r} = \sum_{j=0}^{\infty} p^{(j,r)}\, \theta^j/j!\, r^j \tag{4.13}$$

Subtracting (4.13) from (4.12) we get

$$0 = \sum \{ F(j)(1-\theta)^{-1/r} - 1 \}\, p^{(j,r)}\, \theta^j/j!\, r^j \tag{4.14}$$

Thus, $F(j)$ has to be equal to $(1-\theta)^{1/r}$. Therefore,

$$f(n) = (1-\theta)^{1/r}\, 1^{(n,r)}\, \theta^n/n!\, r^n$$

$$= \frac{\Gamma(n+1/r)}{n!\, \Gamma(1/r)}\, \theta^{1/r}\, (1-\theta)^n,$$

which is $nb(n;\ 1/c,\ \theta)$.

5. CHARACTERIZATIONS BASED ON CONDITIONAL DISTRIBUTIONS

We now prove the following theorems.

Theorem 4: If X and Y are two independent r.v's which follow the NPD (2.9) with parameters $(a/c,p)$ and $(b/c,p)$ respectively, then the conditional distribution of X given X+Y=n, is the PD (2.1).

Proof: By definition of conditional probability, and by independence of r.v's X and Y, we have

$$P(X=x|X+Y=n) = \frac{P(X=x)\ P(Y=n-x)}{P(X+Y=n)}$$

$$= \frac{\dfrac{a^{(x,c)}\ p^{a/c}\ q^{x}}{x!\ c^{x}}\ \dfrac{b^{(n-x,c)}\ p^{b/c}\ q^{n-x}}{(n-x)!\ c^{n-x}}}{\dfrac{(a+b)^{(n,c)}\ p^{(a+b)/c}\ q^{n}}{n!\ c^{n}}}$$

$$= \binom{n}{x} a^{(x,c)}\ b^{(n-x,c)} / (a+b)^{(n,c)} .$$

The converse of this theorem is also true which is given below.

Theorem 5: Let X and Y be two independent discrete r.v's such that $P(X=x|X+Y=n)$ is given by the PD (2.1), then X and Y must have NPD's with parameters $(a/c,p)$ and $(b/c,p)$ respectively.

Proof: $P(X=x|X+Y=n) = \binom{n}{x} a^{(x,c)}\ b^{(n-x,c)} / (a+b)^{(n,c)}$

This is of the form $\alpha(x)\ \beta(y)/\gamma(n)$, where $\alpha(x) = a^{(x,c)} / x!$, $\beta(y) = b^{(y,c)}/y!$ and $\gamma(n) = (a+b)^{(n,c)}/n!$

Hence, from a theorem of Janardan (197[illegible]), the p.f.'s of X and Y are, respectively,

$$f(x) = u\ \alpha(x)\ e^{wx} \quad \text{and} \quad g(y) = v\beta(y)\ e^{wy}$$

where u, v and w are some arbitrary constants setting $e^{w} = (q/c)$, we have

$$f(x) = u\ a^{(x,c)}(q/c)^{x}/x!,\ g(y) = v b^{(y,c)}(q/c)^{y}/y!$$

using the fact $\sum_{x=0}^{\infty} f(x) = 1 = \sum_{y=0}^{\infty} g(y)$, the normalizing constants u and v are seen to be equal to $p^{a/c}$ and $p^{b/c}$ respectively. Thus X and Y are IBD's.

Theorem 6: If a non-negative integral r.v. N is subdivided into two components X and Y such that the conditional distribution $P(X=k, Y=n-k \mid N=n)$ is an IBD as given in (3.1), then the r.v's X and Y are independent if, and only if, N has an IBD.

Proof: The joint probability of X and Y becomes

$$P(X=k, Y=n-k) = \binom{n}{k} \frac{a^{(k,c)}\, b^{(n-k,c)}}{(a+b)^{(n,c)}}\, P(N=n). \qquad (5.1)$$

If N has an IBD (2.2), then one can always write it in the form

$$P(N=n) = (a+b)^{(n,c)}\, p^{(a+b)/c} (1-p)^n / n!\, c^n, \qquad (5.2)$$

where $0 < p < 1$, $a > 0$, and $b > 0$. Substituting the value of $P(N=n)$ in (5.1), gives $P(X=k, Y=n-k) = P(X=k)\, P(Y=n-k)$, where $P(X=k)$ and $P(Y=n-k)$ are both given by IBD's with parameters $(a/c, \theta)$ and $(b/c, \theta)$ respectively.

Conversely, if X and Y are independent r.v's such that $P(X=k \mid N=n)$ is an IBD, then by theorem 6, X and Y are both IBD's and $N=X+Y$ is also an IBD.

Theorem 7: If X and Y are two independent r.v's defined on non-negative integers such that $P(X=x) = f(x) > o$, $\sum_{x=o}^{\infty} f(x) = 1$ and $P(Y=y) = g(y) > o$, $\sum_{y=o}^{\infty} g(y) = 1$, and further if, $p_n + q_n = 1$,

$$P(X=k \mid X+Y=n) = \binom{n}{k} p_n^{(k,r)}\, q_n^{(n-k.r)} / 1^{(n,r)}, \quad k \le n \qquad (5.3)$$

$$= 0, \qquad k > n$$

then (i) p_n is independent of n and equals a constant p for all values

(ii) X and Y must have NBP's with parameters $(p/r,\theta)$ and $(q/r,\theta)$ respectively.

<u>Proof</u>: Since X and Y are independent r.v's,

$$P(X=k\,|\,X+Y=n) = f(k)g(n-k) \Big/ \sum_{k=o}^{n} f(k)g(n-k),$$

which is given by (5.3) for all values of $k \leq n$. For $n > o$ and $o \leq k \leq n$, this provides the functional relation

$$\frac{f(k)g(n-k)}{f(k-1)g(n-k+1)} = \frac{n-k+1}{k} \frac{p_n^{(k,r)} q_n^{(n-k,r)}}{p_n^{(k-1,r)} q_n^{(n-k+1,r)}} \qquad (5.4)$$

Replacing k and n in (5.4) by (k+1) and (n+1) respectively gives

$$\frac{f(k+1)g(n-k)}{f(k)g(n-k+1)} = \frac{n-k+1}{k+1} \frac{p_{n+1}^{(k+1,r)} q_{n+1}^{(n-k,r)}}{p_{n+1}^{(k,r)} q_{n+1}^{(n-k+1,r)}} \qquad (5.5)$$

Dividing (5.5) by (5.4) we get

$$\frac{f(k+1)\, f(k-1)}{f^2(k)} = \frac{k}{k+1} \frac{p_{n+1}^{(k+1,r)}\, p_n^{(k-1,r)}\, q_n^{(n-k,r)}}{p_n^{(k,r)}\, p_{n+1}^{(k,r)}\, q_{n+1}^{(n-k,r)}} \qquad (5.6)$$

The L.H.S. of (5.6) is independent of n and thus R.H.S. of (5.6) must be independent of n. Therefore $p_{n+1} = p_n = p$ for all n.

Substituting $k = 1,2,\ldots,(n-1)$ and multiplying together we get the recurrence relation:

$$f(n) = \alpha\,(p+(n-1)r)\, f(n-1)/n, \qquad (5.7)$$

where $\alpha = f(1)/f(o)$. The relation (5.7) is true for all $n = 1,2,\ldots$ and thus continued substitution yields

$$f(n) = p^{(n,r)}\, \alpha^n\, f(o)/n! \qquad (5.8)$$

Since $\sum f(n) = 1$ the series (5.8) must converge to unity. Let the unknown quantity α be equal to (θ/r), $r \neq o$, $o < \theta < 1$. Then the sum of the

series (5.[illegible]) becomes [illegible] $f(o)$.

Thus,

and $$f(o) = (1-[illegible])^{p/r},$$

$$f(x) = \frac{\Gamma(x+p/r)}{x!\,\Gamma(p/r)} \; (1-\theta)^{p/r}\, \theta^{x}. \qquad (5.9)$$

Similarly, by setting k=1 in (5.5) we can easily get

$$g(n) = \theta(q+(n-1)r)\; g(n-1)/n. \qquad (5.10)$$

Therefore $g(n) = q^{(n,r)}\,\theta^{n}\, g(o)/n!$ and the hypothesis $\sum g(n) = 1$ will yield $g(o) = (1-\theta)^{q/r}$. Hence, the r.v. Y must also be an [illegible] with parameters $(q/r, \theta)$.

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